Intro to AI,
Autumn, 2025

FIRST-ORDER LOGIC

Faculty of DS & AI
Autumn semester, 2025

Trong-Nghia Nguyen

Business AI Lab

2025-10

Content

- **Limitations of propositional logic**
- **First-Order Logic**

First-Order Logic

Limitations of propositional logic

- Limited expressive power
- Cannot fully describe the world of objects, their properties, and the relationships between them

Example:

1. *Nam is a student of National Economics University.*
2. *All students of National Economics University study "Data science."*
3. *Since Nam is a student of National Economics University, Nam studies "Data science".*

*In propositional logic, **(3) cannot be inferred from (1) and (2).***

First-Order Logic

FOL / Predicate logic

- FOL is an extension of propositional logic.
- FOL are statements that contain variables.

  - **NEU_Student (x):** *x is a student of NEU*
    - **NEU_Student (Male):** *Nam is a student of NEU*
  - **Studies_DataScience (x):** *x studies Data Science*
    - **Studies_DataScience (Male):** *Nam studies Data Science*
  - ∀ x: NEU _Student(x) → Studies_ DataScience (x): All students of NEU study Data Science

- In predicate logic, we can prove that:

{ NEU_Student (Male) , ∀x:NEU_Student(x) → Studies_DataScience (x) } →* *Studies_DataScience (Male)*

First-Order Logic

FOL / Predicate logic

Aspect	Propositional Logic (PL)	First-Order Logic (FOL)
Basic Unit	Proposition (a statement that is either true or false)	Predicate (expresses a property or relation among objects)
Structure	No variables or quantifiers	Includes variables (x, y, …) and quantifiers (∀, ∃)
Domain of Discourse	Does not refer to individual objects	Refers explicitly to objects within a domain
Representation Power	Limited – can only express facts as true/false	More expressive – can describe properties, relationships, and rules
Quantification	Not supported	Supported through universal (∀) and existential (∃) quantifiers
Example	"If it rains, the road is wet." → Rain → Wet	"If it rains, every road becomes wet." → ∀x [Rain(x) → Wet(x)]
Knowledge Level	Deals with overall facts or statements	Deals with internal structure of facts and objects
Inference	Works with whole propositions only	Allows reasoning about individuals and their relations
Applications	Boolean circuits, simple rule-based systems	Knowledge representation, semantic web, reasoning in AI
Expressiveness	Lower	Higher (PL ⊂ FOL)

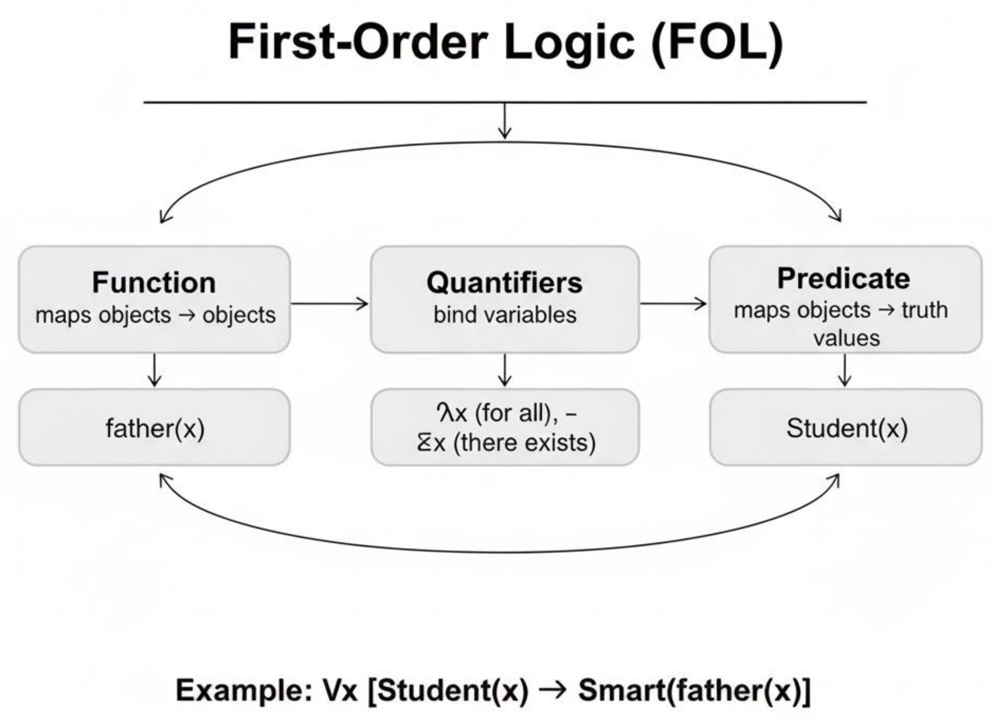
First-Order Logic

Syntax

- **Constants:** represent specific objects in a particular domain
  - *a, b, c, Mai, Nam, John,...*
- **Variables:** represent general or unspecified objects.
  - *x,y,z,u,v,w ,...*
- **Functions:** represent transformations or mappings between objects.
  - *best_friend (x), father(x), distance(x,y), ...*
- **Predicates:** represent relationships between objects or properties of objects
  - *friend(x,y), father(x,y), love(x,y), good(x),...*
- **Quantifiers :**
  - *Universal quantifier (∀): for all*
  - *Existential quantifier (∃): there exists*

First-Order Logic

Syntax



- **Functions** map objects to other objects (e.g., father(x))
- **Quantifiers** (∀, ∃) bind variables to express universal or existential statements.
- **Predicates** map objects to truth values (e.g., Student(x)).

These components work together to represent logical relationships, as shown in the example:

∀x [Student(x) → Smart(father(x))], meaning "For all x, if x is a student, then x's father is smart."

First-Order Logic

Quantum Logic "For All"

- ∀ (variable 1 , variable 2 ,…,variable n): <proposition>

- For example:

    - An is everyone's friend
        - *$\forall x$: friend(An,x)*
    - All pet owners love animals
        - *$\forall x, \forall y$: rear(x,y) ∧ animal(y) → love_animal(x)*
    - All students of FDA are hardworking
        - *$\forall x$: FDA_student(x)→hardworking(x)*

First-Order Logic

Quantum Logic "For All"

- The proposition (∀x:P) is true if and only if P is true for every object in the domain of discourse

- Examples:

    - "*An is a friend of everyone*" is true if and only if:
        - *friend(An, Nam) ∧ friend (An, Mai) ∧ friend(An, Son)*

    - All students of FDA are hardworking when:
        - *FDA_student(Van)→hardworking(Van) ∧ FDA_student(Hung)→hardworking(Hung) ∧ FDA_student(Binh)→hardworking(Binh)*

First-Order Logic

Quantum Logic "Existence"

- The proposition (∃x :P) is true if and only if P is true for at least one object in the domain of discourse
- Examples:
  - There is a person who is An's friend.
    - *$\exists x$: friend(An,x)*
  - There is a person who raises an animal but does not love animals.
    - *$\exists x, \exists y$: rear(x,y) $\wedge$ animal(y) $\wedge$ $\neg$love_animal(x)*
  - There is a student of FDA and hardworking
    - *$\exists x$: FDA_student(x) $\wedge$ hardworking(x)*
  - There is a student who studies all subjects in AI.
    - *$\exists x$, $\forall y$: student(x) $\wedge$ learn(x,y) $\wedge$ AI_subjects(y)*

First-Order Logic

Quantum Logic “Existence”

- Examples:
  - An is a friend of **at least one person** when
    - *friend(An, Nam) v friend (An, Mai) v friend(An, Son)*
  - **At least one** FDA student is hardworking when:
    - *FDA_student(Van)→hardworking(Van) V FDA_student(Hung)→hardworking(Hung) V FDA_student(Binh)→hardworking(Binh)*

First-Order Logic

Properties of quantum logic

- Permutation
  - $(\forall x\ \forall y) \equiv (\forall y\ \forall x)$
  - $(\exists x\ \exists y) \equiv (\exists y\ \exists x)$
  - However, **($\exists$x $\forall$y) is not equivalent to ($\forall$x $\exists$y)**
    - $\exists x\ \forall y\ Know(x,y)$: there exists a person who knows all fields
    - $\forall x\ \exists y\ Know(x,y)$: there exists a field that everyone knows

- Put quantizations into each predicate
  - $\forall x\ ((G(x) \wedge H(x)) \equiv (\forall x\ G(x)) \wedge (\forall x\ H(x))$
  - $\exists x\ ((G(x) \vee H(x)) \equiv (\exists x\ G(x)) \vee (\exists x\ H(x))$

- Eliminate $\forall$
  - $\forall x\ G(x) \equiv G(x)$

First-Order Logic

Properties of quantum logic

- Rename variable
  - $\forall x\, G(x) \equiv \forall y\, G(y)$
  - $\exists x\, G(x) \equiv \exists y\, G(y)$
- Remove ¬
  - $\neg(\forall x\, G(x)) \equiv \exists x\, (\neg G(x))$
  - $\neg(\exists x\, G(x)) \equiv \forall x\, (\neg G(x))$
- Consequence: each quantum (∀,∃) can be represented by the other quantum
  - $[\forall x\, friend(An,x)] \equiv [\neg\exists x\, \neg friend(An,x)]$

    An is a friend of everyone, which is logically equivalent to saying that there exists no one who is not An's friend
  - $[\exists x\, friend(An,x)] \equiv [\neg\forall x\, \neg friend(An,x)]$

    At least one person is friends with An — that is, not everyone isn't friends with An

First-Order Logic

Proving in FOL

- Similar to propositional logic, **value assignments** are also used
- Proof method by **resolution rule**
  - Assume that the conclusion is false.
  - ***Convert the formulas into conjunctive normal form (CNF)*** and separate them into individual clauses.
  - Resolution: find two clauses to resolve until a contradiction is reached
    - Clause i contains *G(x, …)*
    - Clause j contains *¬G(A, …)*
  - By applying the substitution [x | A], the two clauses i and j are unified, and G is eliminated
- Mechanisms for selecting pairs of clauses for resolution:
  - Rigid exhaustive search
  - Flexible search

First-Order Logic

Proving in FOL

Example 1

- Given the knowledge base (premises):

1. Anyone who deceives others is considered a cheater

2. Whether intentionally or unintentionally, if anyone ever agree with someone to deceive others, you will be considered a cheater.

3. Because of being timid, there may be people who, in certain circumstances, agree with others to deceive.

- **Conclusion:**
  Hence, it can be inferred that not everyone who is considered a cheater is not timid.

First-Order Logic

Proving in FOL

Example 1

- Declaring predicates
  - *deceiver(x): x is a person who deceives others*
  - *cheater(x): x is a cheater*
  - *agree(x,y): x agrees with y*
  - *timid(x): x is a timid person*

- Knowledge representation
  1. $deceiver(x) \rightarrow cheater(x)$
  2. $agree(x,y) \wedge cheater(y) \rightarrow cheater(x)$
  3. $\exists x\ (timid(x) \wedge \exists y(agree(x,y) \wedge deceiver(y)))$

- Conclusion $\neg(\forall x\ cheater(x) \rightarrow \neg timid(x))$

Proving in FOL

Example 1

- Add $\neg$(Conclusion) is $(\forall x\ cheater(x) \rightarrow \neg timid(x))$
- Standardize and separate
  1. $\neg deceiver(x) \lor cheater(x)$
  2. $\neg agree(x,y) \lor \neg cheater(y) \lor cheater(x)$
  3. $timid(A)$
  4. $agree(A,B)$
  5. $deceiver(B)$
  6. $\neg cheater(x) \lor \neg timid(x)$
- Apply the solution method

Proving in FOL

Example 2

Given the following knowledge base

1. Hung **likes** all kinds of **food.**
2. Apples are food
3. Chicken is food
4. Anything that people **eat** and don't get harmed is food
5. Phong ate peanuts and still **lived.**
6. Lan eats whatever Phong eats.

Use the solution method to:

- Prove that Hung likes peanuts

First-Order Logic

Proving in FOL

Example 3

Knowledge base:

1. All dogs bark at night.
2. Anyone who has a cat has no mice in their house.
3. Anyone who has trouble sleeping does not keep any animal that barks at night.
4. Mrs. Binh has either a cat or a dog.

Conclusion:

If Mrs. Binh has trouble sleeping, then there are no mice in her house.

First-Order Logic

Proving in FOL

Example 4

Given the following statements:

- John owns a dog.
- All people who own dogs are animal lovers.
- Animal lovers do not kill animals.

Prove: John does not kill animals.

Proving in FOL

Example 5

Given:

- Thuy is a girl.
- An is a boy.
- Girls have longer hair than boys.

Prove: Thuy’s hair is longer than An’s hair.

Thank you!

You're now ready to explore the exciting world of AI!